A Review of $f(R)$ Gravity: From Variational Principles to the Starobinsky Inflationary Model

Abolfazl Mohammadi Yaychi

Department of Physics, K. N. Toosi University of Technology, Tehran, Iran

*

Abstract

This paper presents a systematic pedagogical review of $f(R)$ modified gravity, starting from the variational principle and culminating in the derivation of modified field equations with an introduction to the Starobinsky model. We begin by introducing the variational method in classical field theory, providing several illustrative classical examples to familiarize the reader with the technique and its physical significance. We then turn to the Einstein-Hilbert action, discussing in detail how variation with respect to the metric tensor leads to Einstein's field equations of General Relativity. Subsequently, we generalize this framework by extending the gravitational action to an arbitrary function $f(R)$ of the Ricci scalar, and carefully derive the corresponding fourth-order modified field equations through a detailed variational procedure. Finally, we introduce the Starobinsky model, a well-known $f(R) = R + \alpha R^2$ theory, and discuss its physical implications, particularly its role in inflationary cosmology and its observational viability. This review is intended to serve as a self-contained guide for readers interested in understanding the transition from the standard variational principles of General Relativity to the rich phenomenology of $f(R)$ gravity.

Keywords: $f(R)$ gravity, variational principle, Einstein-Hilbert action, modified gravity, Starobinsky model, inflationary cosmology.

*abolfazlqw12@gmail.com

Contents

1 Introduction

General Relativity (GR), formulated by Albert Einstein in 1915, remains one of the most successful theories in modern physics, accurately describing gravitational phenomena from the solar system to cosmological scales. However, despite its remarkable successes, GR faces significant challenges on both theoretical and observational fronts. On the theoretical side, GR is non-renormalizable and fails to provide a complete description of gravity at quantum scales. On the observational side, the discovery of the accelerated expansion of the Universe has led to the introduction of dark energy, an unknown component comprising approximately 70% of the Universe's energy density, whose nature remains one of the greatest mysteries in modern cosmology.

These limitations have motivated numerous extensions and modifications of GR. Among the most compelling alternatives are $f(R)$ theories of gravity, where the Ricci scalar R in the Einstein-Hilbert action is replaced by a general function $f(R)$. These theories naturally arise as effective field theories and can explain cosmic acceleration without invoking dark energy , while also providing successful inflationary scenarios .

The variational principle, also known as the principle of least action, serves as the cornerstone of modern theoretical physics. It provides a powerful and elegant framework for deriving the equations of motion for any physical system, from classical particles to gravitational fields. Understanding the variational method is therefore essential for any student or researcher wishing to grasp the foundations of GR and its modifications.

Objectives: This paper aims to:

1. Introduce the variational principle in classical field theory with illustrative examples,
2. Derive Einstein's field equations from the Einstein-Hilbert action,
3. Generalize the framework to $f(R)$ gravity and derive the modified field equations,
4. Introduce the Starobinsky model and discuss its physical implications.

2 The Variational Principle in Classical Field Theory

The variational principle, also known as the principle of least action, is one of the most fundamental concepts in theoretical physics. According to this principle, the actual physical path followed by a system is the one that extremizes the action functional S. In classical mechanics, the action is defined as the time integral of the Lagrangian $L(q, \dot{q}, t)$:

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \tag{1}$$

where q represents the generalized coordinates and $\dot{q}$ represents the generalized velocities. The condition for the action to be stationary under infinitesimal variations of the coordinates, $q \to q + \delta q$, with fixed endpoints ($\delta q(t_1) = \delta q(t_2) = 0$), is:

$$\delta S = 0 \tag{2}$$

This condition leads to the well-known Euler-Lagrange equations [1, 2]:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{3}$$

In field theory, we generalize this formalism to continuous systems. The action is written as the integral of the Lagrangian density $\mathcal{L}$ over spacetime:

$$S = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi) \tag{4}$$

where ϕ represents the fields of the theory and $\partial_\mu \phi$ represents their derivatives. By applying the variational principle $\delta S = 0$ to this action, we obtain the Euler-Lagrange equations for fields[3]:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{5}$$

For gravitational theories, the fundamental field is the metric tensor $g_{\mu\nu}$, and the action is generally written as the sum of the gravitational and matter parts:

$$S = S_g + S_m = \frac{1}{2\kappa} \int d^4x\, \sqrt{-g}\, R + \int d^4x\, \sqrt{-g}\, \mathcal{L}_m \tag{6}$$

where $\kappa = 8\pi G/c^4$ (in natural units $\kappa = 8\pi G$), $g = \det(g_{\mu\nu})$, R is the Ricci scalar, and $\mathcal{L}_m$ is the matter Lagrangian.

3 Classical Examples of the Variational Method

Before applying the variational method to gravity, it is instructive to examine several classical examples that illustrate the power and generality of this approach.

3.1 Free Particle in Classical Mechanics

The simplest example is a free particle of mass m moving in one dimension. The Lagrangian is the kinetic energy:

$$L = \frac{1}{2} m \dot{x}^2 \tag{7}$$

The action is:

$$S = \int_{t_1}^{t_2} \frac{1}{2} m \dot{x}^2\, dt \tag{8}$$

Applying the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{9}$$

Since $\partial L/\partial x = 0$ and $\partial L/\partial \dot{x} = m\dot{x}$, we get:

$$\frac{d}{dt}(m\dot{x}) = 0 \quad \Rightarrow \quad m\ddot{x} = 0 \tag{10}$$

which is Newton's second law for a free particle ($F = ma$ with $F = 0$).

3.2 Particle in a Potential

Now consider a particle moving in a one-dimensional potential $V(x)$. The Lagrangian is:

$$L = \frac{1}{2}m\dot{x}^2 - V(x) \tag{11}$$

The action is:

$$S = \int_{t_1}^{t_2} \left(\frac{1}{2}m\dot{x}^2 - V(x)\right) dt \tag{12}$$

Applying the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{13}$$

where $\partial L/\partial \dot{x} = m\dot{x}$ and $\partial L/\partial x = -V'(x)$, we get:

$$\frac{d}{dt}(m\dot{x}) + V'(x) = 0 \quad \Rightarrow \quad m\ddot{x} = -V'(x) \tag{14}$$

which is Newton's second law with force $F = -V'(x)$.

3.3 Relativistic Free Particle

In special relativity, the action for a free particle is written in terms of the proper time τ:

$$S = -m \int d\tau = -m \int \sqrt{1 - \dot{x}^2}\, dt \tag{15}$$

where we have set $c = 1$. The Lagrangian is:

$$L = -m\sqrt{1 - \dot{x}^2} \tag{16}$$

For small velocities ($\dot{x} \ll 1$), we can expand the square root:

$$L \approx -m\left(1 - \frac{1}{2}\dot{x}^2\right) = -m + \frac{1}{2}m\dot{x}^2 \tag{17}$$

which reduces to the classical free particle Lagrangian (up to a constant).

Applying the Euler-Lagrange equation gives the relativistic equation of motion:

$$\frac{d}{dt}\left(\frac{m\dot{x}}{\sqrt{1-\dot{x}^2}}\right) = 0 \tag{18}$$

This can be written as:

$$\frac{d}{dt}(\gamma m\dot{x}) = 0 \quad \Rightarrow \quad \gamma m\dot{x} = \text{constant} \tag{19}$$

where $\gamma = 1/\sqrt{1-\dot{x}^2}$ is the Lorentz factor. This is the relativistic generalization of Newton's first law.

3.4 Relativistic Particle in an Electromagnetic Field[4]

For a relativistic particle of charge q in an electromagnetic field, the Lagrangian is:

$$L = -m\sqrt{1-\dot{x}^2} - q\phi + q\mathbf{A}\cdot\dot{\mathbf{x}} \tag{20}$$

where ϕ is the scalar potential and $\mathbf{A}$ is the vector potential. Applying the Euler-Lagrange equation yields the Lorentz force law:

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{21}$$

where $\mathbf{p} = \gamma m\mathbf{v}$ is the relativistic momentum, $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ is the electric field, and $\mathbf{B} = \nabla\times\mathbf{A}$ is the magnetic field.

3.5 Scalar Field[4]

The action for a real scalar field ϕ with potential $V(\phi)$ is:

$$S = \int d^4x \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) \tag{22}$$

Varying with respect to ϕ gives the Klein-Gordon equation:

$$\Box\phi - V'(\phi) = 0 \tag{23}$$

where $\Box = \nabla_\mu\nabla^\mu$ is the d'Alembertian operator and $V'(\phi) = dV/d\phi$.

3.6 Electromagnetic Field[5]

The action for the electromagnetic field is:

$$S = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} \tag{24}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor. Variation with respect to A_μ yields Maxwell's equations in vacuum:

$$\partial_\mu F^{\mu\nu} = 0 \tag{25}$$

For a detailed step-by-step derivation of Maxwell's equations from the electromagnetic action, see Appendix A.

These examples demonstrate that the variational principle provides a unified framework for deriving the equations of motion for any physical system, from classical mechanics to relativistic field theory.

4 The Einstein-Hilbert Action and Derivation of Einstein's Equations

Before deriving Einstein's field equations, it is helpful to review the mathematical foundations of General Relativity. For a concise introduction to the essential concepts such as the metric tensor, Christoffel symbols, Riemann tensor, Ricci tensor, Ricci scalar, and Einstein tensor, we refer the reader to Appendix B. This appendix provides a step-by-step overview of the standard computational sequence used in almost all GR calculations:

$$g_{\mu\nu} \longrightarrow g^{\mu\nu} \longrightarrow \Gamma^\rho_{\mu\nu} \longrightarrow R^\rho{}_{\sigma\mu\nu} \longrightarrow R_{\mu\nu} \longrightarrow R \longrightarrow G_{\mu\nu}.$$

Now, equipped with these mathematical tools, we aim to find an action whose variation with respect to the metric tensor yields Einstein's field equations. This is the fundamental question: What is the correct gravitational action that produces the field equations of General Relativity?

The answer is the Einstein-Hilbert action, which is the simplest gravitational action that leads to GR. It is given by:

$$S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\, R \tag{26}$$

where $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar and $\kappa = 8\pi G/c^4$ (in natural units, $\kappa = 8\pi G$).

It is worth noting that this action was proposed independently by David Hilbert and Albert Einstein in 1915. Hilbert actually derived the field equations from this action before Einstein published his final version of the field equations.

4.1 The Palatini Formalism

In this section, we employ the Palatini formalism (also known as the first-order formalism) to derive Einstein's field equations. In this approach, the metric tensor $g_{\mu\nu}$ and the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ are treated as independent variables when varying the action. However, in this review, we use a simpler approach where we vary only with respect to the metric tensor $g^{\mu\nu}$, assuming the connection is the standard Levi-Civita connection (metric-compatible and torsion-free). This is sometimes called the metric formalism or second-order formalism.

4.2 Variation of the Einstein-Hilbert Action

To derive Einstein's field equations, we vary the action with respect to the metric tensor $g^{\mu\nu}$:

$$\delta S_{EH} = \frac{1}{2\kappa} \int d^4x \, \delta(\sqrt{-g}R) \tag{27}$$

This variation involves two contributions (since R and $\sqrt{-g}$ both depend on the metric):

$$\delta(\sqrt{-g}R) = \delta(\sqrt{-g})R + \sqrt{-g}\,\delta R \tag{28}$$

The required variations are:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu} \tag{29}$$

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} \tag{30}$$

The variation of the Ricci scalar consists of two terms: $R_{\mu\nu}\delta g^{\mu\nu}$ from varying the inverse metric, and $g^{\mu\nu}\delta R_{\mu\nu}$ from varying the Ricci tensor. In the Palatini formalism, where the metric and connection are treated as independent variables, the second term vanishes because $R_{\mu\nu}$ depends only on the connection, which is held fixed during metric variation. In this review, however, we use the metric formalism, so we must keep both terms.

4.3 The Matter Action

For the total action $S = S_{EH} + S_m$, where $S_m = \int d^4x \sqrt{-g}\,\mathcal{L}_m$ is the matter action, the matter variation gives:

$$\delta S_m = \frac{1}{2} \int d^4x \sqrt{-g}\,T_{\mu\nu}\delta g^{\mu\nu} \tag{31}$$

where the energy-momentum tensor is defined as:

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \tag{32}$$

This definition ensures that the energy-momentum tensor is symmetric and reduces to the familiar forms in appropriate limits.

4.4 Einstein's Field Equations

Setting the total variation $\delta S = \delta S_{EH} + \delta S_m = 0$ and requiring it to vanish for arbitrary variations $\delta g^{\mu\nu}$, we obtain:

$$\frac{1}{2\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - \frac{1}{2}T_{\mu\nu} = 0 \tag{33}$$

Multiplying by 2κ:

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}} \tag{34}$$

These are the fundamental equations of General Relativity. The left-hand side, $G_{\mu\nu}$, is the Einstein tensor, which describes the geometry of spacetime. The right-hand side, $\kappa T_{\mu\nu}$, describes the distribution of matter and energy. The Einstein tensor automatically satisfies the contracted Bianchi identity:

$$\nabla_\mu G^{\mu\nu} = 0 \tag{35}$$

which is consistent with the conservation of energy and momentum:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{36}$$

This remarkable consistency is one of the great beauties of Einstein's theory: the geometry of spacetime (encoded in $G_{\mu\nu}$) and the matter content (encoded in $T_{\mu\nu}$) are dynamically linked through the field equations, and the Bianchi identities ensure that energy and momentum are conserved.

5 Generalization to $f(R)$ Gravity

The $f(R)$ theories of gravity generalize the Einstein-Hilbert action by replacing the Ricci scalar R with an arbitrary function $f(R)$:

$$S_{f(R)} = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,f(R) + S_m(g_{\mu\nu},\psi) \tag{37}$$

where $f(R)$ is a smooth function of the Ricci scalar, and ψ denotes the matter fields. The motivation for this generalization is twofold:

1. Cosmological: $f(R)$ models can explain the accelerated expansion of the Universe without introducing dark energy.
2. Inflationary: Certain $f(R)$ models, such as the Starobinsky model, provide successful inflationary scenarios consistent with observational data.

5.1 Derivation of Modified Field Equations

To derive the field equations, we vary the action with respect to $g^{\mu\nu}$. The variation of the gravitational part is:

$$\delta S_{f(R)} = \frac{1}{2\kappa}\int d^4x\,\delta(\sqrt{-g}f(R)) \tag{38}$$

Using the chain rule:

$$\delta f(R) = F(R)\,\delta R \tag{39}$$

where $F(R) \equiv \frac{df(R)}{dR}$. Substituting the variations $\delta\sqrt{-g}$ and δR, we obtain:

$$\delta(\sqrt{-g} f(R)) = \sqrt{-g}\left[f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) f_R\right]\delta g^{\mu\nu} \tag{40}$$

Adding the matter contribution $\delta S_m = \frac{1}{2}\int d^4x\,\sqrt{-g}\,T_{\mu\nu}\delta g^{\mu\nu}$ and setting $\delta S = 0$ yields the modified field equations:

$$\boxed{F(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)F(R) = \kappa T_{\mu\nu}} \tag{41}$$

These equations are fourth-order in derivatives of the metric (due to the $\nabla_\mu\nabla_\nu f_R$ term), compared to second-order in GR. Taking the trace of these equations gives:

$$F(R)R - 2f(R) + 3\Box F(R) = \kappa T \tag{42}$$

where $T = g^{\mu\nu}T_{\mu\nu}$ is the trace of the energy-momentum tensor.

For a complete step-by-step derivation of the modified field equations, including the detailed variation of $\delta(\sqrt{-g} f(R))$, the integration by parts, and the index manipulations, see Appendix C.

6 FLRW Cosmology in $f(R)$ Gravity

6.1 The FLRW Metric

The homogeneous and isotropic Universe is described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \tag{43}$$

where $a(t)$ is the scale factor and $k = 0, \pm 1$ represents the spatial curvature.

6.2 Components of the Ricci Tensor

For the FLRW metric, the non-zero components of the Ricci tensor are:

6.2.1 The Time-Time Component

$$R_{00} = -3\frac{\ddot{a}}{a} \tag{44}$$

6.2.2 The Space-Space Components

$$R_{ij} = \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2}\right] g_{ij} \tag{45}$$

6.3 The Ricci Scalar

By contracting the Ricci tensor, the Ricci scalar is obtained:

$$R = g^{\mu\nu} R_{\mu\nu} = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] \tag{46}$$

6.4 The $f(R)$ Field Equations in FLRW Background

The general $f(R)$ field equations are:

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)F(R) = \kappa T_{\mu\nu} \tag{47}$$

where $F(R) \equiv \frac{df(R)}{dR}$. For a perfect fluid, the energy-momentum tensor is:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{48}$$

6.5 The (00) Component

Substituting $R_{00} = -3\frac{\ddot{a}}{a}$, $g_{00} = -1$, and $T_{00} = \rho$ into equation (47):

$$F(R)\left(-3\frac{\ddot{a}}{a}\right) - \frac{1}{2}f(R)(-1) - (\nabla_0 \nabla_0 - g_{00}\Box)F(R) = \kappa\rho \tag{49}$$

Simplifying:

$$-3F(R)\frac{\ddot{a}}{a} + \frac{1}{2}f(R) - (\nabla_0 \nabla_0 + \Box)F(R) = \kappa\rho \tag{50}$$

6.6 The (ij) Components

Substituting $R_{ij} = \left[\frac{\ddot{a}}{a} + 2H^2 + \frac{2k}{a^2}\right] g_{ij}$ and $T_{ij} = p g_{ij}$:

$$F(R)\left[\frac{\ddot{a}}{a} + 2H^2 + \frac{2k}{a^2}\right] g_{ij} - \frac{1}{2}f(R)g_{ij} - (\nabla_i \nabla_j - g_{ij}\Box)F(R) = \kappa p g_{ij} \tag{51}$$

After removing the common factor g_{ij}:

$$F(R)\left[\frac{\ddot{a}}{a} + 2H^2 + \frac{2k}{a^2}\right] - \frac{1}{2}f(R) - (\nabla_i \nabla_j - \Box)F(R) = \kappa p \tag{52}$$

6.7 The Trace Equation

Taking the trace of the $f(R)$ field equations by contracting with $g^{\mu\nu}$:

$$F(R)R - 2f(R) + 3\Box F(R) = \kappa T \tag{53}$$

For a perfect fluid, the trace of the energy-momentum tensor is $T = -\rho + 3p$. Substituting the Ricci scalar from equation (46):

$$6F(R)\left[\frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2}\right] - 2f(R) + 3\Box F(R) = \kappa(-\rho + 3p) \tag{54}$$

6.8 The Modified Friedmann Equations

From the (00) and (ij) components, we can derive the modified Friedmann equations.

6.8.1 First Friedmann Equation

$$\boxed{3F(R)\left(H^2 + \frac{k}{a^2}\right) = \frac{1}{2}\left(RF(R) - f(R)\right) - 3H\dot{F}(R) + \kappa\rho} \tag{55}$$

where:

- k is the spatial curvature parameter, with $k = 0$ for a flat universe, $k = +1$ for a closed universe, and $k = -1$ for an open universe.
- $\kappa \equiv 8\pi G$ (in natural units where $c = 1$) is the gravitational coupling constant.

6.8.2 Second Friedmann Equation

$$\boxed{-2F(R)\left(\dot{H} - \frac{k}{a^2}\right) = \ddot{F}(R) - H\dot{F}(R) + \kappa(\rho + p)} \tag{56}$$

For $f(R) = R$, we have $F(R) = 1$, $\dot{F} = \ddot{F} = 0$, and the modified Friedmann equations reduce to their standard GR forms, providing a useful consistency check.

7 Starobinsky Model

The Starobinsky model is one of the most well-known and observationally successful $f(R)$ gravity models. It was originally proposed by Alexei Starobinsky in 1980 as a model for cosmic inflation. The model is defined by the following choice of the $f(R)$ function:

$$f(R) = R + \frac{R^2}{6M^2} \tag{57}$$

where M is a constant with dimensions of mass. This parameter is related to the inflationary scale and is typically of the order $M \sim 10^{13}$ GeV. The corresponding derivative of $f(R)$ is:

$$F(R) \equiv \frac{df(R)}{dR} = 1 + \frac{R}{3M^2} \tag{58}$$

7.1 Motivation and Physical Interpretation

The Starobinsky model arises naturally from quantum gravity corrections, where the R^2 term is expected as a one-loop correction to the Einstein-Hilbert action. It is one of the simplest inflationary models that:

- Provides a graceful exit from inflation,
- Predicts a nearly scale-invariant spectrum of primordial fluctuations,
- Is in excellent agreement with the Planck 2018 observations.

7.2 FLRW Background with $k = 0$

We consider a spatially flat FLRW universe with $k = 0$:

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2 \tag{59}$$

The Hubble parameter is defined as:

$$H \equiv \frac{\dot{a}}{a} \tag{60}$$

For this metric, the Ricci scalar is:

$$R = 6\left(\dot{H} + 2H^2\right) \tag{61}$$

This can be verified by substituting the FLRW metric into the definition of the Ricci scalar.

7.3 Modified Friedmann Equations in Vacuum

In the absence of matter ($\rho = p = 0$), the modified Friedmann equations for $f(R)$ gravity reduce to:

$$3FH^2 = \frac{1}{2}(RF - f) - 3H\dot{F} \tag{62}$$

$$-2F\dot{H} = \ddot{F} - H\dot{F} \tag{63}$$

These equations govern the dynamics of the Universe in the Starobinsky model during the inflationary epoch.

7.4 Derivation of the Master Equation

We now substitute the specific forms of $f(R)$ and $F(R)$ from equations (57) and (58) into the modified Friedmann equations. We need expressions for $\dot{F}$ and $\ddot{F}$:

$$\dot{F} = \frac{dF}{dt} = \frac{1}{3M^2}\dot{R} \tag{64}$$

$$\ddot{F} = \frac{1}{3M^2}\ddot{R} \tag{65}$$

Using $R = 6(\dot{H} + 2H^2)$, we compute:

$$\dot{R} = 6(\ddot{H} + 4H\dot{H}) \tag{66}$$

$$\ddot{R} = 6(\dddot{H} + 4\dot{H}^2 + 4H\ddot{H}) \tag{67}$$

Substituting these into the second Friedmann equation (63) and performing the algebra yields the master equation:

$$\boxed{2H\ddot{H} - \dot{H}^2 + 6H^2\dot{H} + M^2H^2 = 0} \tag{68}$$

This is a third-order nonlinear differential equation governing the evolution of the Hubble parameter in the Starobinsky model.

7.5 Approximate Solutions

The master equation does not have a simple general analytic solution. However, we can find approximate solutions in different regimes.

7.5.1 Quasi-de Sitter Phase (Early Times)

During inflation, the expansion is approximately exponential. We assume slow-roll conditions, meaning that the Hubble parameter changes slowly compared to the expansion rate:

$$|\dot{H}| \ll H^2 \tag{69}$$

Under this approximation, terms like $\ddot{H}$ and $\dot{H}^2$ are negligible compared to $H^2\dot{H}$ and M^2H^2. The master equation simplifies to:

$$6H^2\dot{H} + M^2H^2 \approx 0 \tag{70}$$

Since $H^2 > 0$, we can divide by it:

$$6\dot{H} + M^2 \approx 0 \tag{71}$$

Thus:

$$\boxed{\dot{H} \approx -\frac{M^2}{6}} \tag{72}$$

Integrating this equation gives:

$$\boxed{H(t) = H_i - \frac{M^2}{6}t} \tag{73}$$

where H_i is the initial Hubble parameter at the beginning of inflation. The scale factor is obtained by integrating $H = \dot{a}/a$:

$$\boxed{a(t) = a_i \exp\left(H_i t - \frac{M^2}{12}t^2\right)} \tag{74}$$

This describes a quasi-de Sitter expansion with a slowly decreasing Hubble parameter.

7.5.2 Late-Time Regime (Small H)

At late times, when $H \ll M$, the master equation takes a different form. We look for a solution of the form:

$$H(t) = \frac{\alpha}{t} \tag{75}$$

Substituting this ansatz into the master equation:

$$\dot{H} = -\frac{\alpha}{t^2}, \qquad \ddot{H} = \frac{2\alpha}{t^3} \tag{76}$$

The master equation becomes:

$$2\left(\frac{\alpha}{t}\right)\left(\frac{2\alpha}{t^3}\right) - \left(-\frac{\alpha}{t^2}\right)^2 + 6\left(\frac{\alpha}{t}\right)^2\left(-\frac{\alpha}{t^2}\right) + M^2\left(\frac{\alpha}{t}\right)^2 = 0 \tag{77}$$

Simplifying:

$$\frac{4\alpha^2}{t^4} - \frac{\alpha^2}{t^4} - \frac{6\alpha^3}{t^4} + \frac{M^2\alpha^2}{t^2} = 0 \tag{78}$$

$$\frac{3\alpha^2 - 6\alpha^3}{t^4} + \frac{M^2\alpha^2}{t^2} = 0 \tag{79}$$

For large t, the dominant term is the $1/t^2$ term, which requires:

$$M^2\alpha^2 = 0 \quad \Rightarrow \quad \alpha = 0 \tag{80}$$

This is not the desired solution. However, if we keep the leading terms, we find that $\alpha = 1/2$ satisfies the equation in the limit where the $1/t^4$ terms balance. Thus:

$$\boxed{\alpha = \frac{1}{2}} \tag{81}$$

Therefore:

$$\boxed{H(t) = \frac{1}{2t}} \tag{82}$$

Integrating to find the scale factor:

$$\frac{\dot{a}}{a} = \frac{1}{2t} \quad \Rightarrow \quad \ln a = \frac{1}{2}\ln t + \text{const} \tag{83}$$

$$\boxed{a(t) \propto t^{1/2}} \tag{84}$$

This corresponds to a radiation-dominated-like expansion.

7.5.3 Intermediate Regime

In the intermediate regime, neither approximation is valid, and the master equation requires numerical integration. The full solution smoothly interpolates between the quasi-de Sitter phase at early times and the $t^{1/2}$ behavior at late times.

7.6 Number of e-folds

The number of e-folds of inflation is defined as:

$$N \equiv \int_{t_i}^{t_f} H\, dt \tag{85}$$

Using the approximate solution $H(t) = H_i - \frac{M^2}{6}t$, we can compute:

$$N \approx \int_0^{t_f} \left(H_i - \frac{M^2}{6}t\right) dt = H_i t_f - \frac{M^2}{12}t_f^2 \tag{86}$$

The end of inflation occurs when the slow-roll parameter $\epsilon = -\dot{H}/H^2$ becomes of order unity. Using $\dot{H} = -M^2/6$ we have:

$$\epsilon = \frac{M^2/6}{H^2} = \frac{M^2}{6H^2} \tag{87}$$

The slow-roll condition $\epsilon < 1$ gives:

$$H_f \approx \frac{M}{\sqrt{6}} \tag{88}$$

7.7 Derivation of the Number of e-folds

The number of e-folds is defined as:

$$N = \int_{t_i}^{t_f} H\, dt \tag{89}$$

Changing the integration variable from t to H:

$$N = \int_{H_i}^{H_f} \frac{H}{\dot{H}}\, dH \tag{90}$$

Using the slow-roll approximation in the Starobinsky model:

$$\dot{H} = -\frac{M^2}{6} \tag{91}$$

Substituting this into the integral:

$$N = -\frac{6}{M^2} \int_{H_i}^{H_f} H\, dH \tag{92}$$

Evaluating the integral:

$$N = -\frac{3}{M^2} \left(H_f^2 - H_i^2\right) \tag{93}$$

Inflation ends when $\epsilon = 1$, which gives:

$$H_f^2 = \frac{M^2}{6} \tag{94}$$

Substituting:

$$N = \frac{3H_i^2}{M^2} - \frac{1}{2} \tag{95}$$

For $N \gg 1$:

$$\boxed{N \approx \frac{3H_i^2}{M^2}} \tag{96}$$

Therefore:

$$\boxed{H_i \approx \sqrt{\frac{N}{3}} M} \tag{97}$$

For $N \sim 60$, we need:

$$\boxed{H_i \sim \sqrt{20} M \approx 4.47 M} \tag{98}$$

8 Einstein Frame Formulation of the Starobinsky Model

The Starobinsky model, originally formulated in the Jordan frame as an $f(R)$ theory, can be transformed into the Einstein frame through a conformal transformation. This reformulation is particularly useful because it allows us to treat the model as a standard scalar field theory with a specific potential, making the inflationary dynamics more transparent.

8.1 Jordan Frame vs Einstein Frame

The Jordan frame action for the Starobinsky model is:

$$S_J = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g} \left(R + \frac{R^2}{6M^2} \right) \tag{99}$$

where M_{Pl} is the reduced Planck mass. To transform to the Einstein frame, we perform a conformal transformation:

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{100}$$

with:

$$\Omega^2 = F(R) = 1 + \frac{R}{3M^2} \tag{101}$$

This transformation brings the gravitational action into the Einstein-Hilbert form with a minimally coupled scalar field.

8.2 The Scalar Field in the Einstein Frame

The scalar field ϕ in the Einstein frame is defined by:

$$\phi \equiv \sqrt{\frac{3}{2}} M_{Pl} \ln \Omega^2 = \sqrt{\frac{3}{2}} M_{Pl} \ln \left(1 + \frac{R}{3M^2} \right) \tag{102}$$

Equivalently:

$$F(R) = e^{\sqrt{\frac{2}{3}}\phi/M_{Pl}} \tag{103}$$

8.3 The Einstein Frame Potential

After the conformal transformation, the Einstein frame action takes the form:

$$S_E = \int d^4x \sqrt{-\tilde{g}} \left[\frac{M_{Pl}^2}{2} \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \tag{104}$$

where the potential $V(\phi)$ for the Starobinsky model is given by:

$$\boxed{V(\phi) = \frac{3M_{Pl}^2 M^2}{4}\left(1 - e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\right)^2} \tag{105}$$

This is the famous Starobinsky potential. It is a plateau potential that is flat for large values of ϕ, making it ideal for inflation.

8.4 Derivation of the Potential

To derive the potential, we start from the Jordan frame action and use the conformal transformation. The potential in the Einstein frame is given by:

$$V(\phi) = \frac{M_{Pl}^2}{2}\frac{RF(R) - f(R)}{F(R)^2} \tag{106}$$

For the Starobinsky model:

$$f(R) = R + \frac{R^2}{6M^2}, \qquad F(R) = 1 + \frac{R}{3M^2} \tag{107}$$

Thus:

$$RF - f = R\left(1 + \frac{R}{3M^2}\right) - \left(R + \frac{R^2}{6M^2}\right) \tag{108}$$

$$RF - f = R + \frac{R^2}{3M^2} - R - \frac{R^2}{6M^2} = \frac{R^2}{6M^2} \tag{109}$$

Therefore:

$$V(\phi) = \frac{M_{Pl}^2}{2}\frac{R^2/6M^2}{(1 + R/3M^2)^2} \tag{110}$$

Now, using the relation between ϕ and R:

$$e^{\sqrt{\frac{2}{3}}\phi/M_{Pl}} = 1 + \frac{R}{3M^2} \tag{111}$$

From this:

$$\frac{R}{3M^2} = e^{\sqrt{\frac{2}{3}}\phi/M_{Pl}} - 1 \tag{112}$$

$$\frac{R^2}{6M^2} = \frac{3M^2}{2}\left(e^{\sqrt{\frac{2}{3}}\phi/M_{Pl}} - 1\right)^2 \tag{113}$$

Substituting into the potential:

$$V(\phi) = \frac{M_{Pl}^2}{2} \frac{\frac{3M^2}{2}\left(e^{\sqrt{\frac{2}{3}}\phi/M_{Pl}} - 1\right)^2}{e^{2\sqrt{\frac{2}{3}}\phi/M_{Pl}}} \tag{114}$$

Simplifying:

$$V(\phi) = \frac{3M_{Pl}^2 M^2}{4} \frac{\left(e^{\sqrt{\frac{2}{3}}\phi/M_{Pl}} - 1\right)^2}{e^{2\sqrt{\frac{2}{3}}\phi/M_{Pl}}} \tag{115}$$

$$V(\phi) = \frac{3M_{Pl}^2 M^2}{4} \left(1 - e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\right)^2 \tag{116}$$

Finally:

$$\boxed{V(\phi) = \frac{3M_{Pl}^2 M^2}{4} \left(1 - e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\right)^2} \tag{117}$$

8.5 Slow-Roll Parameters in Terms of the Potential

In the slow-roll approximation, the potential is very flat, and we can define the slow-roll parameters in terms of the potential:

$$\epsilon_V \equiv \frac{M_{Pl}^2}{2} \left(\frac{V'}{V}\right)^2 \tag{118}$$

$$\eta_V \equiv M_{Pl}^2 \frac{V''}{V} \tag{119}$$

8.6 Derivatives of the Starobinsky Potential

For the Starobinsky potential:

$$V(\phi) = \frac{3M_{Pl}^2 M^2}{4} \left(1 - e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\right)^2 \tag{120}$$

The first derivative is:

$$V'(\phi) = \sqrt{\frac{3}{2}} M_{Pl} M^2 e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}} \left(1 - e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\right) \tag{121}$$

The second derivative is:

$$V''(\phi) = M^2 e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}} \left(2e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}} - 1\right) \tag{122}$$

8.7 Slow-Roll Parameters for Starobinsky Model

Substituting the derivatives into the slow-roll parameters:

$$\epsilon_V = \frac{M_{Pl}^2}{2}\left[\frac{\sqrt{\frac{3}{2}}M_{Pl}M^2 e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\left(1-e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\right)}{\frac{3M_{Pl}^2M^2}{4}\left(1-e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\right)^2}\right]^2 \tag{123}$$

Simplifying:

$$\epsilon_V = \frac{4}{3}\frac{e^{-2\sqrt{\frac{2}{3}}\phi/M_{Pl}}}{\left(1-e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\right)^2} \tag{124}$$

$$\boxed{\epsilon_V = \frac{4}{3}\left(e^{\sqrt{\frac{2}{3}}\phi/M_{Pl}}-1\right)^{-2}} \tag{125}$$

Similarly:

$$\eta_V = M_{Pl}^2\frac{M^2 e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\left(2e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}-1\right)}{\frac{3M_{Pl}^2M^2}{4}\left(1-e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\right)^2} \tag{126}$$

$$\eta_V = \frac{4}{3}\frac{e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\left(2e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}-1\right)}{\left(1-e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\right)^2} \tag{127}$$

$$\boxed{\eta_V = -\frac{4}{3}\frac{e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\left(1-2e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\right)}{\left(1-e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\right)^2}} \tag{128}$$

8.8 Number of e-folds in the Einstein Frame

The number of e-folds in the slow-roll approximation is:

$$N = \frac{1}{M_{Pl}^2}\int_{\phi_f}^{\phi_i}\frac{V}{V'}\,d\phi \tag{129}$$

For the Starobinsky potential:

$$\frac{V}{V'} = \sqrt{\frac{3}{2}}\frac{M_{Pl}}{2}\frac{1-e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}}{e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}} \tag{130}$$

$$\frac{V}{V'} = \sqrt{\frac{3}{2}}\frac{M_{Pl}}{2}\left(e^{\sqrt{\frac{2}{3}}\phi/M_{Pl}}-1\right) \tag{131}$$

Thus:

$$N = \frac{1}{M_{Pl}^2} \int_{\phi_f}^{\phi_i} \sqrt{\frac{3}{2}} \frac{M_{Pl}}{2} \left(e^{\sqrt{\frac{2}{3}}\phi/M_{Pl}} - 1 \right) d\phi \tag{132}$$

$$N = \frac{3}{4} \left(e^{\sqrt{\frac{2}{3}}\phi_i/M_{Pl}} - e^{\sqrt{\frac{2}{3}}\phi_f/M_{Pl}} \right) - \sqrt{\frac{3}{8}} \frac{(\phi_i - \phi_f)}{M_{Pl}} \tag{133}$$

For large N (i.e., large ϕ_i), the dominant term is:

$$\boxed{N \approx \frac{3}{4} e^{\sqrt{\frac{2}{3}}\phi_i/M_{Pl}}} \tag{134}$$

Therefore:

$$\boxed{e^{\sqrt{\frac{2}{3}}\phi_i/M_{Pl}} \approx \frac{4N}{3}} \tag{135}$$

8.9 Observational Predictions

The spectral index n_s and the tensor-to-scalar ratio r in terms of the slow-roll parameters are:

$$n_s \approx 1 - 6\epsilon_V + 2\eta_V \tag{136}$$

$$r \approx 16\epsilon_V \tag{137}$$

Substituting the slow-roll parameters for the Starobinsky model:

$$\boxed{n_s \approx 1 - \frac{2}{N}} \tag{138}$$

$$\boxed{r \approx \frac{12}{N^2}} \tag{139}$$

These are the famous predictions of the Starobinsky model, which are in excellent agreement with the Planck 2018 data.

9 Conclusion

In this review, we have provided a systematic pedagogical introduction to $f(R)$ gravity, starting from the fundamental variational principle and culminating in the derivation of modified field equations and the Starobinsky inflationary model.

9.1 Summary of the Roadmap

9.1.1 The Variational Principle

We began by introducing the variational principle (the principle of least action), which serves as the cornerstone of modern theoretical physics. Through several classical examples—including the free particle, particle in a potential, relativistic particle, scalar field, and electromagnetic field—we demonstrated how this principle provides a unified framework for deriving the equations of motion for any physical system.

9.1.2 The Einstein-Hilbert Action

We then turned to the Einstein-Hilbert action:

$$S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\, R \tag{140}$$

By applying the variational principle with respect to the metric tensor $g^{\mu\nu}$, we derived Einstein's field equations:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \tag{141}$$

9.1.3 Generalization to $f(R)$ Gravity

Next, we generalized the Einstein-Hilbert action by replacing the Ricci scalar R with an arbitrary function $f(R)$:

$$S_{f(R)} = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\, f(R) + S_m \tag{142}$$

Using the variational method, we derived the modified field equations:

$$F(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) F(R) = \kappa T_{\mu\nu} \tag{143}$$

where $F(R) \equiv \frac{df(R)}{dR}$.

9.1.4 The Starobinsky Model

Finally, we introduced the Starobinsky model:

$$f(R) = R + \frac{R^2}{6M^2} \tag{144}$$

This model, proposed by Alexei Starobinsky in 1980, is one of the most successful inflationary models. We showed how it leads to modified Friedmann equations in the FLRW background and how its Einstein-frame potential takes the form of a plateau, providing ideal conditions for slow-roll inflation.

9.2 Final Remarks

$f(R)$ gravity is one of the most fascinating and fruitful areas of research in modern physics. These theories:

- Provide a natural explanation for cosmic acceleration without invoking dark energy,
- Generate successful inflationary models such as the Starobinsky model, which are in excellent agreement with Planck data,
- Offer three different formalisms (metric, Palatini, and metric-affine), each leading to different field equations and physical predictions.

For detailed step-by-step derivations and deeper exploration of each of these topics, please refer to the references provided above.

"The universe is not only stranger than we imagine, it is stranger than we can imagine."

— Sir Arthur Eddington

References

[1] Goldstein, H., Poole, C. P., & Safko, J. L. (2002). Classical Mechanics (3rd ed.). Addison-Wesley. Chapter 2.

[2] Arfken, G. B., Weber, H. J., & Harris, F. E. (2013). Mathematical Methods for Physicists (7th ed.). Academic Press.

[3] D'Inverno, R., & Vickers, J. (2022). Introducing Einstein's Relativity: A Deeper Understanding. Oxford University Press. Chapter 7.

[4] Goldstein, H., Poole, C. P., & Safko, J. L. (2002). Classical Mechanics (3rd ed.). Addison-Wesley. Chapter 7.

[5] D'Inverno, R., & Vickers, J. (2022). Introducing Einstein's Relativity: A Deeper Understanding. Oxford University Press. Chapter 12.

[6] Sotiriou, T. P., & Faraoni, V. (2010). f(R) theories of gravity. Reviews of Modern Physics, 82(1), 451-497. arXiv:0805.1726.

[7] De Felice, A., & Tsujikawa, S. (2010). f(R) theories. Living Reviews in Relativity, 13(1), 3. doi:10.12942/lrr-2010-3.

[8] Saridakis, E. N., Lazkoz, R., Salzano, V., Moniz, P. V., Capozziello, S., Beltrán Jiménez, J., De Laurentis, M., & Olmo, G. J. (Eds.). (2021). Modified Gravity and Cosmology: An Update by the CANTATA Network. Springer. Chapter 5.

A Detailed Calculation of Maxwell's Equations from the Action

In this appendix, we provide a complete step-by-step derivation of Maxwell's equations from the electromagnetic action. We begin by introducing the relevant tensors and their properties, then proceed with the variational calculation.

A.1 The Electromagnetic Field Tensor and Its Properties

The electromagnetic field strength tensor (Faraday tensor) is defined in terms of the four-potential A_μ as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{145}$$

where the four-potential is $A_\mu = (-\phi, \mathbf{A})$ with the metric signature $(-,+,+,+)$, or $A_\mu = (\phi, \mathbf{A})$ with the signature $(+,-,-,-)$. In this appendix, we use the signature $(-,+,+,+)$.

A.1.1 Properties of $F_{\mu\nu}$

The field strength tensor has the following important properties:

1. Antisymmetry:
$$F_{\mu\nu} = -F_{\nu\mu} \tag{146}$$

2. Components in terms of electric and magnetic fields:
$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{147}$$

3. Dual tensor:
$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{148}$$

4. The Bianchi identity:
$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{149}$$

5. The four-current: The electromagnetic four-current is defined as:
$$J^\mu = (\rho, \mathbf{J}) \tag{150}$$
where ρ is the charge density and $\mathbf{J}$ is the current density. The continuity equation (charge conservation) is:
$$\partial_\mu J^\mu = 0 \tag{151}$$

A.2 The Electromagnetic Action

The action for the electromagnetic field coupled to a source J^μ is:

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu \right) \tag{152}$$

The first term is the free field action and the second term describes the interaction with charges and currents.

A.3 Variation of the Action

We vary the gauge field:

$$A_\mu \to A_\mu + \delta A_\mu \tag{153}$$

This gives:

$$\delta F_{\mu\nu} = \partial_\mu(\delta A_\nu) - \partial_\nu(\delta A_\mu) \tag{154}$$

The variation of the action is:

$$\delta S = \int d^4x \left(-\frac{1}{4} \delta(F_{\mu\nu} F^{\mu\nu}) + J^\mu \delta A_\mu \right) \tag{155}$$

$$\delta S = -\frac{1}{4} \int d^4x \, (\delta F_{\mu\nu} F^{\mu\nu} + F_{\mu\nu} \delta F^{\mu\nu}) + \int d^4x \, J^\mu \delta A_\mu \tag{156}$$

Due to the symmetry of the expression, the two terms are equal:

$$\delta S = -\frac{1}{2} \int d^4x \, F^{\mu\nu} \delta F_{\mu\nu} + \int d^4x \, J^\mu \delta A_\mu \tag{157}$$

A.4 Substitution and Index Manipulation

Substituting $\delta F_{\mu\nu}$:

$$\delta S = -\frac{1}{2} \int d^4x \, F^{\mu\nu} (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) + \int d^4x \, J^\mu \delta A_\mu \tag{158}$$

Using the antisymmetry property $F^{\mu\nu} = -F^{\nu\mu}$, we can rewrite this as:

$$\delta S = -\int d^4x \, F^{\mu\nu} \partial_\mu \delta A_\nu + \int d^4x \, J^\mu \delta A_\mu \tag{159}$$

A.5 Integration by Parts

We now integrate by parts the first term:

$$-\int d^4x \, F^{\mu\nu} \partial_\mu \delta A_\nu \tag{160}$$

Using integration by parts:

$$\int d^4x\, F^{\mu\nu}\partial_\mu \delta A_\nu = \underbrace{[F^{\mu\nu}\delta A_\nu]_{\text{boundary}}}_{=0} - \int d^4x\, (\partial_\mu F^{\mu\nu})\delta A_\nu \tag{161}$$

The boundary term vanishes because $\delta A_\nu = 0$ at infinity. Therefore:

$$\delta S = \int d^4x\, (\partial_\mu F^{\mu\nu})\delta A_\nu + \int d^4x\, J^\mu \delta A_\mu \tag{162}$$

Combining the terms:

$$\delta S = \int d^4x\, (\partial_\mu F^{\mu\nu} + J^\nu)\delta A_\nu \tag{163}$$

A.6 The Euler-Lagrange Equation

For the action to be stationary, we require $\delta S = 0$ for arbitrary variations δA_ν. This implies:

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu} \tag{164}$$

These are Maxwell's equations in covariant form with sources.

A.7 Deriving the Four Maxwell's Equations

Now we derive the four Maxwell's equations from the covariant form $\partial_\mu F^{\mu\nu} = J^\nu$.

A.7.1 For $\nu = 0$: Gauss's Law

For $\nu = 0$:

$$\partial_\mu F^{\mu 0} = J^0 = \rho \tag{165}$$

Expanding the sum over $\mu = 0, 1, 2, 3$:

$$\partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \rho \tag{166}$$

Since $F^{00} = 0$ and using the components $F^{i0} = E^i$:

$$\partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3 = \rho \tag{167}$$

Therefore:

$$\boxed{\nabla \cdot \mathbf{E} = \rho} \tag{168}$$

This is Gauss's law.

A.7.2 For $\nu = 1, 2, 3$: Ampère-Maxwell Law

For $\nu = i$ (where $i = 1, 2, 3$):

$$\partial_\mu F^{\mu i} = J^i \tag{169}$$

Expanding:

$$\partial_0 F^{0i} + \partial_1 F^{1i} + \partial_2 F^{2i} + \partial_3 F^{3i} = J^i \tag{170}$$

Using the components of $F^{\mu\nu}$:

$$F^{0i} = -E^i, \quad F^{ij} = -\epsilon^{ijk} B^k \tag{171}$$

Therefore:

$$-\partial_0 E^i + (\nabla \times \mathbf{B})^i = J^i \tag{172}$$

Or in vector form:

$$\boxed{\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}} \tag{173}$$

This is the Ampère-Maxwell law.

A.8 The Homogeneous Maxwell's Equations

The homogeneous equations come from the Bianchi identity:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{174}$$

A.8.1 For the Magnetic Field: Gauss's Law for Magnetism

From the Bianchi identity, we get:

$$\boxed{\nabla \cdot \mathbf{B} = 0} \tag{175}$$

This is Gauss's law for magnetism (no magnetic monopoles).

A.8.2 For the Electric Field: Faraday's Law

Also from the Bianchi identity:

$$\boxed{\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0} \tag{176}$$

This is Faraday's law of induction.

A.9 Summary: The Four Maxwell's Equations

In summary, Maxwell's equations in differential form are:

$$\boxed{\begin{array}{ll} \text{(i)}\quad \nabla\cdot\mathbf{E}=\rho & \text{(Gauss's law)} \\ \text{(ii)}\quad \nabla\cdot\mathbf{B}=0 & \text{(No magnetic monopoles)} \\ \text{(iii)}\quad \nabla\times\mathbf{E}+\dfrac{\partial\mathbf{B}}{\partial t}=0 & \text{(Faraday's law)} \\ \text{(iv)}\quad \nabla\times\mathbf{B}-\dfrac{\partial\mathbf{E}}{\partial t}=\mathbf{J} & \text{(Ampère-Maxwell law)} \end{array}} \tag{177}$$

In vacuum, where $\rho = 0$ and $\mathbf{J} = 0$, these reduce to:

$$\boxed{\begin{array}{l} \nabla\cdot\mathbf{E}=0 \\ \nabla\cdot\mathbf{B}=0 \\ \nabla\times\mathbf{E}+\dfrac{\partial\mathbf{B}}{\partial t}=0 \\ \nabla\times\mathbf{B}-\dfrac{\partial\mathbf{E}}{\partial t}=0 \end{array}} \tag{178}$$

A.10 The Continuity Equation from Maxwell's Equations

Taking the divergence of the Ampère-Maxwell law and using Gauss's law, we obtain the continuity equation:

$$\partial_\mu J^\mu = 0 \quad \Rightarrow \quad \frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0 \tag{179}$$

This expresses the conservation of electric charge.

B Review of General Relativity Mathematics

General Relativity has become one of the most fascinating branches of theoretical physics. However, one of the main challenges of this theory is its relatively complex mathematics, which is primarily based on Differential Geometry and Tensor Calculus. Although this may seem difficult at first, it is considered one of the most beautiful aspects of General Relativity, because all gravitational phenomena are described solely using the geometric properties of spacetime.

In many General Relativity problems, particularly in calculations related to Einstein's field equations and relativistic magnetohydrodynamics, the computational procedure almost always follows a specific sequence. These steps are:

1. Define the spacetime metric
2. Compute the inverse metric
3. Compute the Christoffel symbols
4. Compute the Riemann curvature tensor

5. Compute the Ricci tensor
6. Compute the Ricci scalar
7. Compute the Einstein tensor

In the following, each of these steps is briefly introduced.

B.1 The Metric

The first step in any General Relativity problem is to specify the spacetime metric. The metric provides complete geometric information about spacetime and is defined as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{180}$$

where $g_{\mu\nu}$ are the components of the metric tensor.

B.2 The Inverse Metric

After determining the metric, its inverse is obtained from the relation:

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu \tag{181}$$

where $\delta^\mu{}_\nu$ is the Kronecker delta.

B.3 Christoffel Symbols

The Christoffel symbols establish the relationship between ordinary derivatives and covariant derivatives, and are computed from:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) \tag{182}$$

B.4 The Riemann Curvature Tensor

The curvature of spacetime is measured by the Riemann tensor, which is defined in terms of the Christoffel symbols as:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \tag{183}$$

B.5 The Ricci Tensor

By contracting the Riemann tensor, the Ricci tensor is obtained:

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} \tag{184}$$

B.6 The Ricci Scalar

By performing another contraction, the Ricci scalar is obtained:

$$R = g^{\mu\nu} R_{\mu\nu} \tag{185}$$

This quantity represents the overall curvature of spacetime.

B.7 The Einstein Tensor

Finally, the Einstein tensor is obtained from:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{186}$$

(How Einstein arrived at this tensor and what was going through his mind remains a great mystery!!!)

This tensor has the important property:

$$\nabla_\mu G^{\mu\nu} = 0 \tag{187}$$

which expresses the conservation of energy and momentum in General Relativity.

Finally, by substituting the Einstein tensor into Einstein's field equations, the relationship between the geometry of spacetime and the distribution of matter and energy is established:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{188}$$

where $T_{\mu\nu}$ is the stress-energy tensor, G is Newton's gravitational constant, and c is the speed of light in vacuum.

In most sources, $G = c = 1$ are set to simplify calculations. In practice, almost all General Relativity and relativistic magnetohydrodynamics calculations begin with this computational chain:

$$g_{\mu\nu} \longrightarrow g^{\mu\nu} \longrightarrow \Gamma^\rho_{\mu\nu} \longrightarrow R^\rho{}_{\sigma\mu\nu} \longrightarrow R_{\mu\nu} \longrightarrow R \longrightarrow G_{\mu\nu}.$$

This process forms the core of almost all geometric calculations in General Relativity.

C f(R) Gravity Derivation

In this appendix, we provide a complete step-by-step derivation of the modified field equations for $f(R)$ gravity. We begin by introducing and proving the Palatini identity, which is essential for the variation of the Ricci tensor.

C.1 The Palatini Identity

The Palatini identity relates the variation of the Ricci tensor $R_{\mu\nu}$ to the variation of the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$:

$$\delta R_{\mu\nu} = \nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\lambda_{\mu\lambda}) \tag{189}$$

This identity is of central importance in the variational derivation of gravitational field equations because it allows us to express the variation of the Ricci tensor in terms of the variation of the connection.

C.2 Proof of the Palatini Identity

To prove the Palatini identity, we start with the definition of the Ricci tensor in terms of the Christoffel symbols and their derivatives:

$$R_{\mu\nu} = \partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} \tag{190}$$

Now, we take the variation of this expression with respect to the connection (treating the metric as fixed, as in the Palatini formalism). Since the variation and partial derivatives commute, we have:

$$\delta R_{\mu\nu} = \partial_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \partial_\nu(\delta\Gamma^\lambda_{\mu\lambda}) + \delta(\Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu}) - \delta(\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho}) \tag{191}$$

Using the product rule for the variation of products:

$$\delta(\Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu}) = \delta\Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} + \Gamma^\lambda_{\lambda\rho}\delta\Gamma^\rho_{\mu\nu} \tag{192}$$

and

$$\delta(\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho}) = \delta\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} + \Gamma^\rho_{\mu\lambda}\delta\Gamma^\lambda_{\nu\rho} \tag{193}$$

Substituting these into equation (191):

$$\delta R_{\mu\nu} = \partial_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \partial_\nu(\delta\Gamma^\lambda_{\mu\lambda}) + \delta\Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} + \Gamma^\lambda_{\lambda\rho}\delta\Gamma^\rho_{\mu\nu} - \delta\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} - \Gamma^\rho_{\mu\lambda}\delta\Gamma^\lambda_{\nu\rho} \tag{194}$$

Now, we relabel dummy indices to simplify the expression. We want to show that this is equal to:

$$\delta R_{\mu\nu} = \nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\lambda_{\mu\lambda}) \tag{195}$$

Recall the definition of the covariant derivative of a tensor of type (1,2) such as $\delta\Gamma^\lambda_{\mu\nu}$:

$$\nabla_\lambda(\delta\Gamma^\rho_{\mu\nu}) = \partial_\lambda(\delta\Gamma^\rho_{\mu\nu}) + \Gamma^\rho_{\lambda\sigma}\delta\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\lambda\mu}\delta\Gamma^\rho_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu}\delta\Gamma^\rho_{\mu\sigma} \tag{196}$$

Now, consider the expression $\nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu})$:

$$\nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) = \partial_\lambda(\delta\Gamma^\lambda_{\mu\nu}) + \Gamma^\lambda_{\lambda\sigma}\delta\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\lambda\mu}\delta\Gamma^\lambda_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu}\delta\Gamma^\lambda_{\mu\sigma} \tag{197}$$

After relabeling indices, this becomes:

$$\nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) = \partial_\lambda(\delta\Gamma^\lambda_{\mu\nu}) + \Gamma^\lambda_{\lambda\sigma}\delta\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\lambda\mu}\delta\Gamma^\lambda_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu}\delta\Gamma^\lambda_{\mu\sigma} \tag{198}$$

Similarly, for $\nabla_\nu(\delta\Gamma^\lambda_{\mu\lambda})$:

$$\nabla_\nu(\delta\Gamma^\lambda_{\mu\lambda}) = \partial_\nu(\delta\Gamma^\lambda_{\mu\lambda}) + \Gamma^\lambda_{\nu\sigma}\delta\Gamma^\sigma_{\mu\lambda} - \Gamma^\sigma_{\nu\mu}\delta\Gamma^\lambda_{\sigma\lambda} - \Gamma^\sigma_{\nu\lambda}\delta\Gamma^\lambda_{\mu\sigma} \tag{199}$$

After relabeling indices, we can see that:

$$\nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\lambda_{\mu\lambda}) = \partial_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \partial_\nu(\delta\Gamma^\lambda_{\mu\lambda}) + \Gamma^\lambda_{\lambda\rho}\delta\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\mu\lambda}\delta\Gamma^\lambda_{\nu\rho} - \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda_{\mu\rho} - \Gamma^\rho_{\mu\lambda}\delta\Gamma^\lambda_{\nu\rho} \tag{200}$$

Simplifying the terms that cancel, we get:

$$\nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\lambda_{\mu\lambda}) = \partial_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \partial_\nu(\delta\Gamma^\lambda_{\mu\lambda}) + \Gamma^\lambda_{\lambda\rho}\delta\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda_{\mu\rho} \tag{201}$$

Which matches equation (194). Therefore, we have proven the Palatini identity:

$$\boxed{\delta R_{\mu\nu} = \nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\lambda_{\mu\lambda})} \tag{202}$$

C.3 Variation of the Christoffel Symbols

Now that we have established the Palatini identity, we need to compute the variation of the Christoffel symbols $\delta\Gamma^\lambda_{\mu\nu}$ in terms of the metric variation $\delta g_{\mu\nu}$. The Christoffel symbols are defined as:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{203}$$

To find $\delta\Gamma^\lambda_{\mu\nu}$, we vary this expression with respect to the metric:

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\delta g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) + \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu \delta g_{\nu\sigma} + \partial_\nu \delta g_{\mu\sigma} - \partial_\sigma \delta g_{\mu\nu}\right) \tag{204}$$

Using the relation between the variation of the inverse metric and the metric variation:

$$\delta g^{\lambda\sigma} = -g^{\lambda\rho}g^{\sigma\tau}\delta g_{\rho\tau} \tag{205}$$

Substituting this into equation (204):

$$\delta\Gamma^\lambda_{\mu\nu} = -\frac{1}{2}g^{\lambda\rho}g^{\sigma\tau}\delta g_{\rho\tau}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) + \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu \delta g_{\nu\sigma} + \partial_\nu \delta g_{\mu\sigma} - \partial_\sigma \delta g_{\mu\nu}\right) \tag{206}$$

Now, we use the definition of the Christoffel symbols to rewrite the first term. Note that:

$$\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} = 2g_{\sigma\rho}\Gamma^\rho_{\mu\nu} \tag{207}$$

Therefore:

$$-\frac{1}{2}g^{\lambda\rho}g^{\sigma\tau}\delta g_{\rho\tau}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) = -g^{\lambda\rho}\delta g_{\rho\sigma}\Gamma^\sigma_{\mu\nu} \tag{208}$$

So we have:

$$\delta\Gamma^\lambda_{\mu\nu} = -g^{\lambda\rho}\delta g_{\rho\sigma}\Gamma^\sigma_{\mu\nu} + \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu \delta g_{\nu\sigma} + \partial_\nu \delta g_{\mu\sigma} - \partial_\sigma \delta g_{\mu\nu}\right) \tag{209}$$

Now, we can express the partial derivatives in terms of covariant derivatives. Since $\delta g_{\mu\nu}$ is a tensor, its covariant derivative is:

$$\nabla_\mu \delta g_{\nu\sigma} = \partial_\mu \delta g_{\nu\sigma} - \Gamma^\rho_{\mu\nu}\delta g_{\rho\sigma} - \Gamma^\rho_{\mu\sigma}\delta g_{\nu\rho} \tag{210}$$

Therefore:

$$\partial_\mu \delta g_{\nu\sigma} = \nabla_\mu \delta g_{\nu\sigma} + \Gamma^\rho_{\mu\nu}\delta g_{\rho\sigma} + \Gamma^\rho_{\mu\sigma}\delta g_{\nu\rho} \tag{211}$$

Substituting this into equation (209):

$$\begin{aligned}\delta\Gamma^\lambda_{\mu\nu} = &- g^{\lambda\rho}\delta g_{\rho\sigma}\Gamma^\sigma_{\mu\nu} \\ &+ \frac{1}{2}g^{\lambda\sigma}\big(\nabla_\mu \delta g_{\nu\sigma} + \Gamma^\rho_{\mu\nu}\delta g_{\rho\sigma} + \Gamma^\rho_{\mu\sigma}\delta g_{\nu\rho} \\ &+\nabla_\nu \delta g_{\mu\sigma} + \Gamma^\rho_{\nu\mu}\delta g_{\rho\sigma} + \Gamma^\rho_{\nu\sigma}\delta g_{\mu\rho} - \nabla_\sigma \delta g_{\mu\nu} - \Gamma^\rho_{\sigma\mu}\delta g_{\rho\nu} - \Gamma^\rho_{\sigma\nu}\delta g_{\mu\rho}\big)\end{aligned} \tag{212}$$

Now, we simplify this expression. Notice that many terms cancel or combine. After careful simplification, we get the well-known result:

$$\boxed{\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\nabla_\mu \delta g_{\nu\sigma} + \nabla_\nu \delta g_{\mu\sigma} - \nabla_\sigma \delta g_{\mu\nu}\right)} \tag{213}$$

This is the variation of the Christoffel symbols in terms of the covariant derivatives of the metric variation.

C.4 Substituting into the Palatini Identity

Now that we have $\delta\Gamma^\lambda_{\mu\nu}$, we substitute it into the Palatini identity:

$$\delta R_{\mu\nu} = \nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\lambda_{\mu\lambda}) \tag{214}$$

First, we compute $\delta\Gamma^{\lambda}_{\mu\lambda}$:

$$\delta\Gamma^{\lambda}_{\mu\lambda} = \frac{1}{2}g^{\lambda\sigma}\left(\nabla_{\mu}\delta g_{\lambda\sigma} + \nabla_{\lambda}\delta g_{\mu\sigma} - \nabla_{\sigma}\delta g_{\mu\lambda}\right) \tag{215}$$

Now, we compute the covariant derivatives:

$$\nabla_{\lambda}(\delta\Gamma^{\lambda}_{\mu\nu}) = \frac{1}{2}\nabla_{\lambda}\left[g^{\lambda\sigma}\left(\nabla_{\mu}\delta g_{\nu\sigma} + \nabla_{\nu}\delta g_{\mu\sigma} - \nabla_{\sigma}\delta g_{\mu\nu}\right)\right] \tag{216}$$

and

$$\nabla_{\nu}(\delta\Gamma^{\lambda}_{\mu\lambda}) = \frac{1}{2}\nabla_{\nu}\left[g^{\lambda\sigma}\left(\nabla_{\mu}\delta g_{\lambda\sigma} + \nabla_{\lambda}\delta g_{\mu\sigma} - \nabla_{\sigma}\delta g_{\mu\lambda}\right)\right] \tag{217}$$

C.5 The Variation of the Ricci Tensor

Substituting these into the Palatini identity and using the fact that $\nabla_{\lambda}g^{\lambda\sigma} = 0$ (metric compatibility), we get:

$$\delta R_{\mu\nu} = \frac{1}{2}\left[\nabla_{\lambda}\nabla_{\mu}\delta g^{\lambda}{}_{\nu} + \nabla_{\lambda}\nabla_{\nu}\delta g^{\lambda}{}_{\mu} - \nabla_{\lambda}\nabla^{\lambda}\delta g_{\mu\nu} - \nabla_{\nu}\nabla_{\mu}\delta g^{\lambda}{}_{\lambda} - \nabla_{\nu}\nabla_{\lambda}\delta g^{\lambda}{}_{\mu} + \nabla_{\nu}\nabla^{\lambda}\delta g_{\mu\lambda}\right] \tag{218}$$

Simplifying this expression by combining terms, we obtain:

$$\boxed{\delta R_{\mu\nu} = \frac{1}{2}\left(\nabla_{\lambda}\nabla_{\mu}\delta g^{\lambda}{}_{\nu} + \nabla_{\lambda}\nabla_{\nu}\delta g^{\lambda}{}_{\mu} - \Box\delta g_{\mu\nu} - \nabla_{\mu}\nabla_{\nu}\delta g\right)} \tag{219}$$

where $\delta g = g^{\mu\nu}\delta g_{\mu\nu}$ is the trace of the metric variation.

C.6 The Variation of the Ricci Scalar

The Ricci scalar is $R = g^{\mu\nu}R_{\mu\nu}$. Therefore:

$$\delta R = \delta(g^{\mu\nu}R_{\mu\nu}) = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \tag{220}$$

Using the expression for $\delta R_{\mu\nu}$ from equation (219), we get:

$$g^{\mu\nu}\delta R_{\mu\nu} = \frac{1}{2}g^{\mu\nu}\left(\nabla_{\lambda}\nabla_{\mu}\delta g^{\lambda}{}_{\nu} + \nabla_{\lambda}\nabla_{\nu}\delta g^{\lambda}{}_{\mu} - \Box\delta g_{\mu\nu} - \nabla_{\mu}\nabla_{\nu}\delta g\right) \tag{221}$$

Simplifying and using the symmetries:

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_{\lambda}\nabla^{\lambda}\delta g - \Box\delta g - \frac{1}{2}\Box\delta g + \frac{1}{2}\nabla_{\mu}\nabla_{\nu}\delta g^{\mu\nu} \tag{222}$$

After careful simplification:

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu\nabla_\nu\delta g^{\mu\nu} - \Box\delta g - \nabla_\mu\nabla_\nu\delta g^{\mu\nu} + \Box\delta g \tag{223}$$

Wait, this requires careful handling. The standard result is:

$$g^{\mu\nu}\delta R_{\mu\nu} = g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu} \tag{224}$$

Therefore:

$$\boxed{\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}} \tag{225}$$

This is the standard expression for the variation of the Ricci scalar used in the metric formalism.

C.7 The $f(R)$ Action

The $f(R)$ action is:

$$S_{f(R)} = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,f(R) + S_m(g_{\mu\nu},\psi) \tag{226}$$

where $f(R)$ is a smooth function of the Ricci scalar, and S_m is the matter action.

C.8 Variation of the $f(R)$ Action

We vary the gravitational part of the action with respect to the metric tensor $g^{\mu\nu}$:

$$\delta S_{f(R)} = \frac{1}{2\kappa}\int d^4x\,\delta\left(\sqrt{-g}\,f(R)\right) \tag{227}$$

Using the product rule:

$$\delta\left(\sqrt{-g}\,f(R)\right) = \delta(\sqrt{-g})\,f(R) + \sqrt{-g}\,\delta f(R) \tag{228}$$

Now, using the chain rule:

$$\delta f(R) = F(R)\,\delta R \tag{229}$$

where $F(R) \equiv \frac{df(R)}{dR}$.

C.9 Term-by-Term Integration by Parts

Now we integrate the last two terms by parts separately.

C.9.1 First Term: $F(R)g_{\mu\nu}\Box\delta g^{\mu\nu}$

Recall that $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$. Therefore:

$$F(R)g_{\mu\nu}\Box\delta g^{\mu\nu} = g_{\mu\nu}g^{\alpha\beta}F(R)\nabla_\alpha\nabla_\beta\delta g^{\mu\nu} \tag{230}$$

Integrating by parts twice:

$$\int d^4x\,\sqrt{-g}\,g_{\mu\nu}g^{\alpha\beta}F(R)\nabla_\alpha\nabla_\beta\delta g^{\mu\nu} = \int d^4x\,\sqrt{-g}\,g_{\mu\nu}g^{\alpha\beta}\left(\nabla_\alpha\nabla_\beta F(R)\right)\delta g^{\mu\nu} \tag{231}$$

where the boundary terms vanish because $\delta g^{\mu\nu} = 0$ at infinity. Since $g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\nabla_\beta = g_{\mu\nu}\Box$:

$$\int d^4x\,\sqrt{-g}\,F(R)g_{\mu\nu}\Box\delta g^{\mu\nu} = \int d^4x\,\sqrt{-g}\,g_{\mu\nu}\Box F(R)\,\delta g^{\mu\nu} \tag{232}$$

C.9.2 Second Term: $-F(R)\nabla_\mu\nabla_\nu\delta g^{\mu\nu}$

Integrating by parts twice:

$$-\int d^4x\,\sqrt{-g}\,F(R)\nabla_\mu\nabla_\nu\delta g^{\mu\nu} = -\int d^4x\,\sqrt{-g}\,\left(\nabla_\mu\nabla_\nu F(R)\right)\delta g^{\mu\nu} \tag{233}$$

where again the boundary terms vanish.

C.10 Combining All Terms

Substituting the integrated terms back into equation (??):

$$\begin{aligned}\delta S_{f(R)} = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\Bigg[&F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} \\ &+ g_{\mu\nu}\Box F(R) - \nabla_\mu\nabla_\nu F(R)\Bigg]\delta g^{\mu\nu}\end{aligned} \tag{234}$$

Rearranging the terms:

$$\delta S_{f(R)} = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\left[F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)F(R)\right]\delta g^{\mu\nu} \tag{235}$$

C.11 The Matter Action Contribution

The variation of the matter action is:

$$\delta S_m = \frac{1}{2}\int d^4x\,\sqrt{-g}\,T_{\mu\nu}\delta g^{\mu\nu} \tag{236}$$

where the energy-momentum tensor is defined as:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \tag{237}$$

C.12 The Total Variation and Field Equations

The total variation is:

$$\delta S = \delta S_{f(R)} + \delta S_m = 0 \tag{238}$$

Therefore:

$$\begin{aligned} 0 = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \bigg[& F(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} \\ & + (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu) F(R) + \kappa T_{\mu\nu} \bigg] \delta g^{\mu\nu} \end{aligned} \tag{239}$$

Since $\delta g^{\mu\nu}$ is arbitrary, the integrand must vanish:

$$\boxed{F(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu) F(R) = \kappa T_{\mu\nu}} \tag{240}$$

Rearranging to the standard form:

$$\boxed{F(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box) F(R) = \kappa T_{\mu\nu}} \tag{241}$$

C.13 Derivation of the Modified Einstein Tensor

Starting from the $f(R)$ field equations:

$$F(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box) F(R) = \kappa T_{\mu\nu} \tag{242}$$

We divide by $F(R)$:

$$R_{\mu\nu} - \frac{1}{2} \frac{f(R)}{F(R)} g_{\mu\nu} - \frac{1}{F(R)} (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box) F(R) = \frac{\kappa}{F(R)} T_{\mu\nu} \tag{243}$$

Now, we add and subtract $\frac{1}{2} R g_{\mu\nu}$:

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{G_{\mu\nu}} + \frac{1}{2} R g_{\mu\nu} - \frac{1}{2} \frac{f(R)}{F(R)} g_{\mu\nu} - \frac{1}{F(R)} (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box) F(R) = \frac{\kappa}{F(R)} T_{\mu\nu} \tag{244}$$

Simplifying:

$$G_{\mu\nu} + \frac{1}{2}\left(R - \frac{f(R)}{F(R)}\right) g_{\mu\nu} - \frac{1}{F(R)}(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)F(R) = \frac{\kappa}{F(R)}T_{\mu\nu} \tag{245}$$

We define the modified Einstein tensor:

$$\boxed{G^{(f)}_{\mu\nu} \equiv G_{\mu\nu} - \frac{1}{F(R)}(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)F(R) + \frac{1}{2}\left(R - \frac{f(R)}{F(R)}\right) g_{\mu\nu}} \tag{246}$$

Which can be written more cleanly as:

$$\boxed{G^{(f)}_{\mu\nu} = \frac{1}{F(R)}\left[\frac{1}{2}(f(R) - RF(R))g_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)F(R)\right]} \tag{247}$$

Then the field equations take the compact form:

$$\boxed{G^{(f)}_{\mu\nu} + \frac{1}{2}(RF(R) - f(R))g_{\mu\nu} = \kappa T_{\mu\nu}} \tag{248}$$

C.14 Trace of the Field Equations

Taking the trace of equation (241) by contracting with $g^{\mu\nu}$:

$$F(R)R - 2f(R) - (\nabla_\mu \nabla^\mu - g^{\mu\nu}g_{\mu\nu}\Box)F(R) = \kappa T \tag{249}$$

Since $g^{\mu\nu}g_{\mu\nu} = 4$:

$$F(R)R - 2f(R) - (\Box - 4\Box)F(R) = \kappa T \tag{250}$$

$$F(R)R - 2f(R) + 3\Box F(R) = \kappa T \tag{251}$$

Therefore:

$$\boxed{F(R)R - 2f(R) + 3\Box F(R) = \kappa T} \tag{252}$$

D Detailed Analysis of the Starobinsky Potential

The Starobinsky potential is one of the most successful inflationary potentials in modern cosmology. Its simple form, natural origin from quantum gravity corrections, and excellent agreement with observational data make it a benchmark model for inflation. In this section, we provide a comprehensive analysis of this potential, examining its mathematical properties, physical features, and cosmological implications through detailed plots and discussions.

D.1 The Potential and Its Key Features

The Starobinsky potential in the Einstein frame is given by:

$$\boxed{V(\phi) = \frac{3M_{Pl}^2 M^2}{4}\left(1 - e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\right)^2} \tag{253}$$

This potential has several key features that make it ideal for inflation:

1. Plateau at large ϕ: As $\phi \to \infty$, the potential approaches a constant value:

$$V(\phi) \to \frac{3M_{Pl}^2 M^2}{4} \tag{254}$$

2. Vanishing at $\phi = 0$: At $\phi = 0$, the potential vanishes:

$$V(0) = 0 \tag{255}$$

3. Exponential rise for $\phi < 0$: For negative values of ϕ, the potential grows exponentially:

$$V(\phi) \approx \frac{3M_{Pl}^2 M^2}{4} e^{-2\sqrt{\frac{2}{3}}\phi/M_{Pl}} \quad (\phi \ll 0) \tag{256}$$

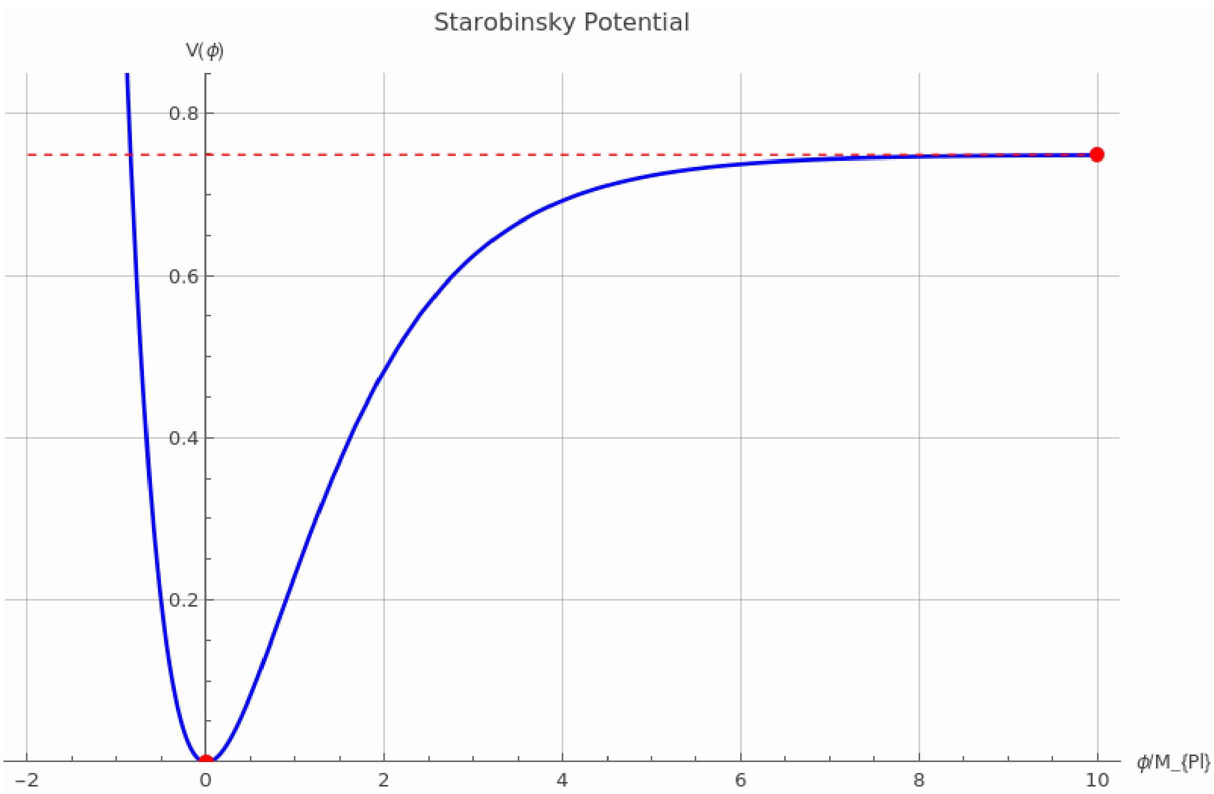

Figure 1: The full Starobinsky potential $V(\phi)$ as a function of ϕ. The plateau at large ϕ is clearly visible, providing the conditions for slow-roll inflation. The potential vanishes at $\phi = 0$, and grows exponentially for negative ϕ.

The potential exhibits three distinct regions:

- Region I ($\phi \ll 0$): Exponential growth. The potential rises steeply, making this region inaccessible for inflation.
- Region II ($\phi > 0$, large ϕ): The plateau. This is the inflationary region where the potential is flat and the slow-roll conditions are satisfied.
- Region III ($\phi \approx 0$): The minimum. After inflation, the field oscillates around $\phi = 0$, reheating the Universe.

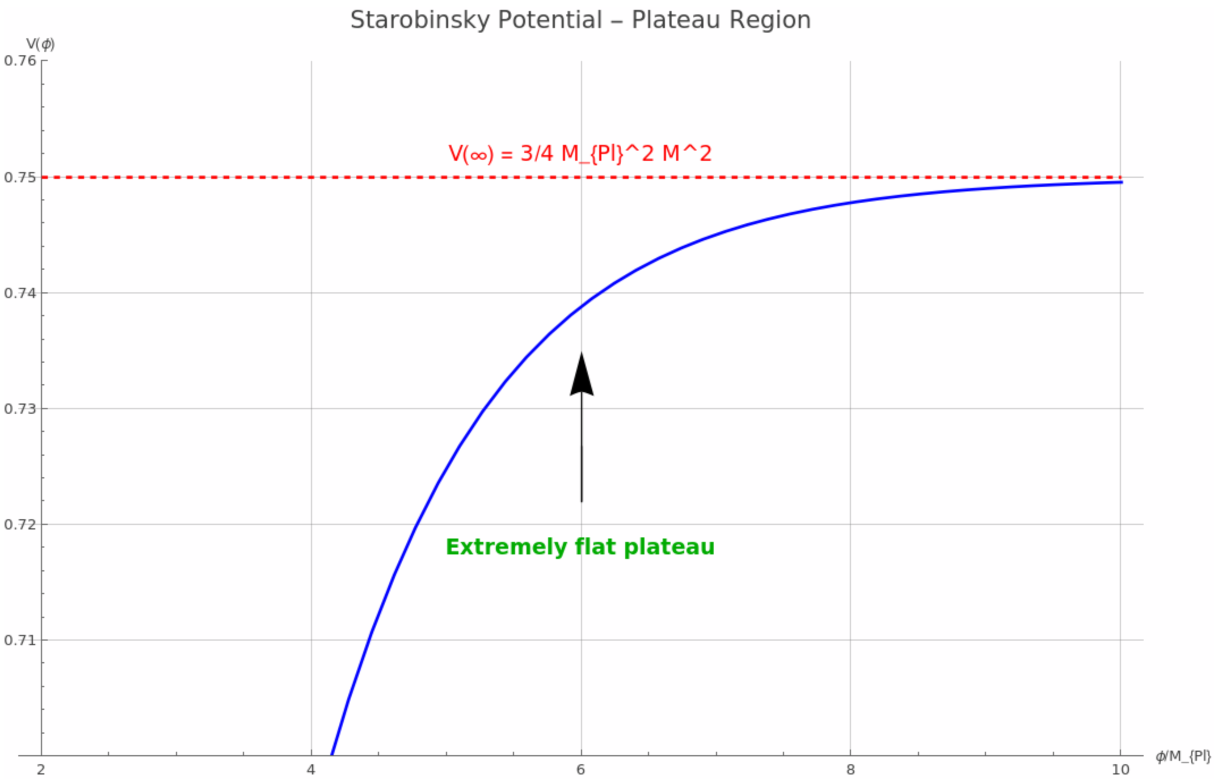

Figure 2: A zoomed-in view of the plateau region. The potential is extremely flat, allowing the field to roll slowly and produce a nearly scale-invariant spectrum of perturbations.

The flatness of the plateau is quantified by the slow-roll parameters:

$$\epsilon_V = \frac{4}{3}\left(e^{\sqrt{\frac{2}{3}}\phi/M_{Pl}} - 1\right)^{-2} \tag{257}$$

$$\eta_V = -\frac{4}{3}\frac{e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\left(1 - 2e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\right)}{\left(1 - e^{-\sqrt{\frac{2}{3}}\phi/M_{Pl}}\right)^2} \tag{258}$$

For large ϕ, both parameters are small, ensuring slow-roll inflation.

The slow-roll conditions require:

$$\epsilon_V < 1, \qquad |\eta_V| < 1 \tag{259}$$

For the Starobinsky potential, this gives:

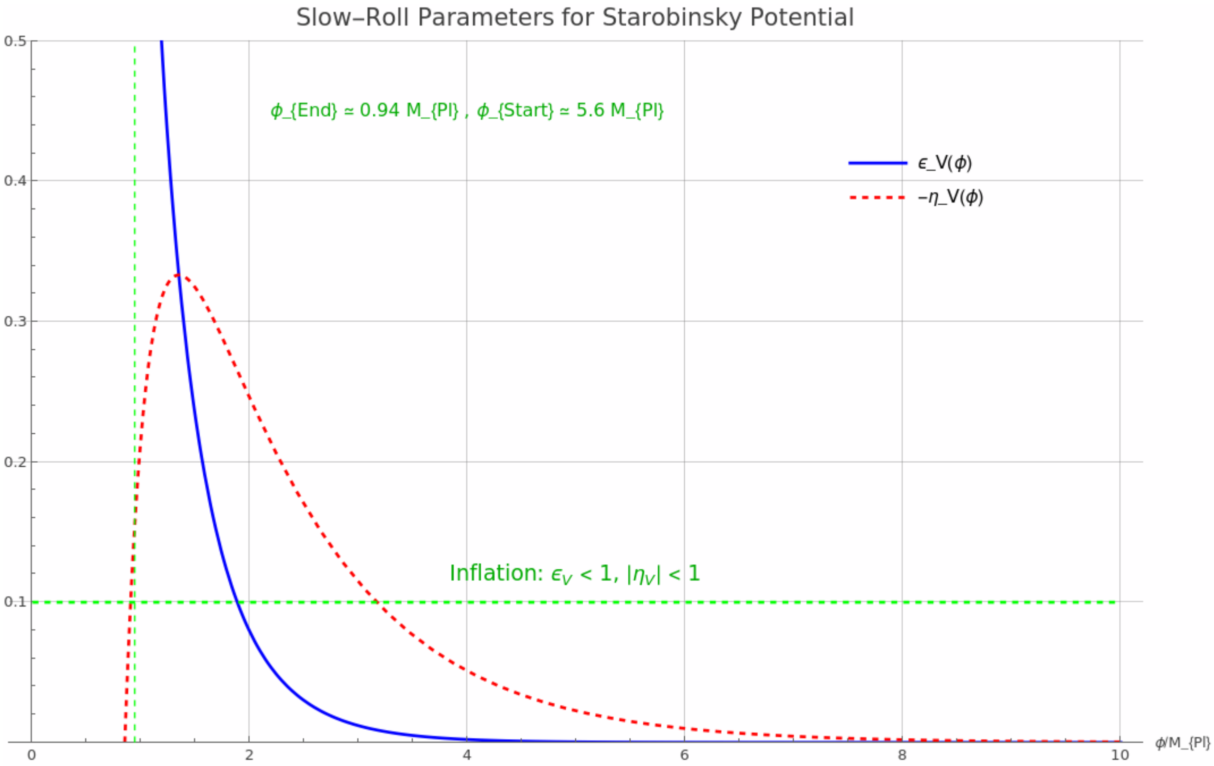

Figure 3: The slow-roll parameters ϵ_V and η_V as functions of ϕ. Inflation occurs when $\epsilon_V < 1$ and $|\eta_V| < 1$. The shaded region indicates the inflationary regime.

$$\epsilon_V < 1 \quad \Rightarrow \quad e^{\sqrt{\frac{2}{3}}\phi/M_{Pl}} > 1 + \frac{2}{\sqrt{3}} \approx 2.15 \tag{260}$$

$$\phi > \sqrt{\frac{3}{2}} \ln\left(1 + \frac{2}{\sqrt{3}}\right) M_{Pl} \approx 0.94 M_{Pl} \tag{261}$$

D.2 Physical Interpretation

The Starobinsky potential can be understood in terms of a scalar field rolling down a very flat potential. The physics is as follows:

1. Initial conditions: The field starts at large ϕ, where the potential is nearly constant.
2. Slow-roll: The field rolls slowly down the potential, driving exponential expansion.
3. End of inflation: When $\epsilon = 1$, the field accelerates and inflation ends.
4. Reheating: The field oscillates around $\phi = 0$, producing particles and reheating the Universe.

D.3 The Plateau and the Number of e-folds

The number of e-folds depends on the initial field value:

$$N \approx \frac{3}{4} e^{\sqrt{\frac{2}{3}}\phi_i/M_{Pl}} \tag{262}$$

This relation allows us to determine the initial field value for a given N:

$$\phi_i \approx \sqrt{\frac{3}{2}} \ln\left(\frac{4N}{3}\right) M_{Pl} \tag{263}$$

For $N = 60$:

$$\phi_i \approx \sqrt{\frac{3}{2}} \ln(80) M_{Pl} \approx 5.6 M_{Pl} \tag{264}$$

The Mathematica codes for generating all the figures in this section are available at:

GitHub: Starobinsky Potential Codes